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BIFURCATIONS OF LIMIT CYCLES FROM CUBIC HAMILTONIAN SYSTEMS WITH A CENTER AND A HOMOCLINIC SADDLE-LOOP

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Abstract

It is proved in this paper that the maximum number of limit cycles of system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = kx - (k+1)x^2 + x^3 + \epsilon(\alpha + \beta x + \gamma x^2)y \end{cases}$$

is equal to two in the finite plane, where $k > \frac{11+\sqrt{33}}{4}$, $0 < |\epsilon| \ll 1$, $|\alpha| + |\beta| + |\gamma| \neq 0$. This is partial answer to the seventh question in [**2**], posed by Arnold.

1. Introduction

Consider the Abelian integral

$$I(h) = \oint_{\Gamma_h} Y(x,y)\,dx - X(x,y)\,dy, \quad h \in \Sigma, \tag{1.1}$$

where $H(x,y)$, $X(x,y)$ and $Y(x,y)$ are real polynomial of x and y, Γ_h is the compact component of $H(x,y) = h$, Σ is the maximal interval of existence of Γ_h. Finding the lowest upper bound for the number of zeros of $I(h)$ is called the weakend Hilber-16th problem [**1**], which is closed related to determining the number of limit cycles of perturbed system

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial y} + \epsilon X(x,y), \\ \dfrac{dy}{dt} = -\dfrac{\partial H}{\partial x} + \epsilon Y(x,y), \end{cases} \tag{1.2}$$

where $0 < |\epsilon| \ll 1$.

This work was done in 1995–1998, when the first author was a Ph.D. student in Peking University.

In particular, suppose

$$H(x,y)=\frac{1}{2}y^2+U(x)=h, \tag{1.3}$$

where $U(x)$ is a real polynomial of x with degree n. In this case, finding the number of zeros of $I(h)$ is one of the ten problems in [**2**]. When $n=3$, this problem was investigated by many authors (e.g. [**7**], [**8**], [**10**], [**11**]). When $n=4$, some results were given by [**5**], [**12**], [**13**], [**16**], [**17**], but this case is far from complete solving. In this paper, we study the case $n=4$ and the Hamiltonian vector field $dH=0$ possesses one center and one homoclinic saddle-loop, which has the following normal form

$$\begin{cases} \dfrac{dx}{dt}=y, \\ \dfrac{dy}{dt}=kx-(k+1)x^2+x^3, \end{cases} \tag{1.4}$$

where $k>2$.

The system (1.4) has the first integral

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{2}kx^2+\frac{1}{3}(k+1)x^3-\frac{1}{4}x^4=h, \tag{1.5}$$

and the phase portrait is shown in Figure 1.1. The closed ovals Γ_h are defined for Hamiltonian values $h\in(\frac{-2k+1}{12},0)$. $H(x,y)=\frac{-2k+1}{12}$ corresponds the center $(1,0)$, $\Gamma_0=\{(x,y)\mid H(x,y)=0,\ 0<x<x_1=\frac{2(k+1)-\sqrt{2(k-2)(2k-1)}}{3}\}$ corresponds the saddle point $(0,0)$ and homoclinic loop. The critical point $(k,0)$ is a saddle.

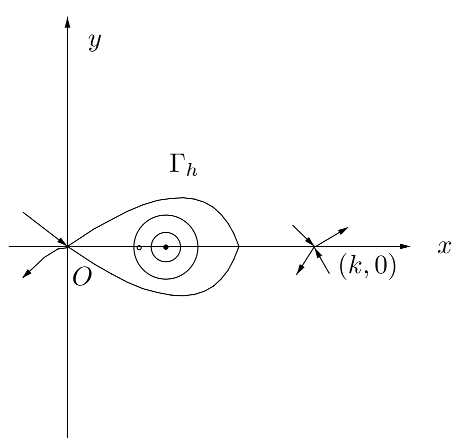

Figure 1.1

Denote

$$I_i(h) = \oint_{\Gamma_h} x^i y\, dx, \quad i = 0, 1, 2, \tag{1.6}$$

$$I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h), \tag{1.7}$$

where the ovals Γ_h, $h \in (\frac{-2k+1}{12}, 0)$, has negative (clockwise) orientation coinciding with the orientation of the vector field (1.4), α, β and γ are arbitrary constants. The central result of this paper is the following theorem:

Theorem 1.1. *The maximum number of limit cycles of the perturbed system*

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = kx - (k+1)x^2 + x^3 + \epsilon(\alpha + \beta x + \gamma x^2)y, \end{cases} \tag{1.8$_\epsilon$}$$

is equal to two in the finite plane, where $k > \frac{11+\sqrt{33}}{4}$, $0 < |\epsilon| \ll 1$, $|\alpha| + |\beta| + |\gamma| \neq 0$.

Corollary 1.2. *For* $k > \frac{11+\sqrt{33}}{4}$, *either* $I(h)$ *vanishes identically or its lowest upper bound of the number of zeros is equal to two, which is partial answer to the seventh problem in* [**2**].

The paper is organized as follows: In section 2, Picard-Fuchs equation satisfied by $I_0(h)$, $I_1(h)$ and $I_2(h)$ is derived and the expansions of $I(h)$ near its endpoints are given, the latter results reveal the connection between the Abelian integrals $I(h)$ and the limit cycles of system $(1.8)_\epsilon$ which tend to the center $(1,0)$ or homoclinic loop of (1.4) as $\epsilon \to 0$. In section 3, instead of estimating the number of zeros of $I(h)$, we will prove that $I''(h)$ has at most two zeros, i.e., $I(h)$ has at most two inflection points in $(\frac{-2k+1}{12}, 0)$, which implies the lowest upper bound of the number of zeros of $I(h)$ does not exceed three in the same interval. Using the fact $\omega(h) = \frac{I_1''(h)}{I_0''(h)}$ satisfies a Riccati equation, we get $\omega'(h) > 0$. Hence, the curve $\widetilde{\Omega} = \{(\omega, \nu) \mid \omega = \frac{I_1''(h)}{I_0''(h)}, \nu = \frac{I_2''(h)}{I_0''(h)}, h \in (\frac{-2k+1}{12}, 0)\}$ can be defined. It is readily seen that the intersection points of line $\alpha + \beta\omega + \gamma\nu = 0$ with $\widetilde{\Omega}$ in $\omega\nu$-plane correspond the zeros of $I''(h)$, which shows that the convexity of $\widetilde{\Omega}$ determinates the number of the zeros of $I''(h)$.

In section 4, we make precise connection between the intersection points of $L\colon \alpha+\beta P+\gamma Q=0$ with the centroid curve $\Omega=\{(P,Q)\mid P=\frac{I_1(h)}{I_0(h)},\ Q=\frac{I_2(h)}{I_1(h)}\}$ on one hand and the zeros of Abelian integral $I(h)$ on the other hand. Finally, the main results of this paper are proved in section 5. Some techniques in section 4 and section 5 are borrowed from [**4**].

Remark. Unfortunately, the techniques we use in this present paper do not fit for the case of $2<k<\frac{11+\sqrt{33}}{4}$. Therefore, throughout this paper, we suppose $k>\frac{11+\sqrt{33}}{4}>4$ unless the opposite is claimed. Some computation in this paper is done by the computer program "Mathematica".

2. Picard-Fuchs equation and the asymptotic expansions of $I(h)$ near its endpoints

In this section we shall derive Picard-Fuchs equation satisfied by $I_i(h)$ and describe the behaviours of $I(h)$ near $h=0$ and $h=\frac{-2k+1}{12}$.

Lemma 2.1. *$I_0(h)$, $I_1(h)$ and $I_2(h)$ satisfy the following Picard-Fuchs equation*

$$(4h\mathbf{E}+\mathbf{S})\mathbf{J}'=\mathbf{N}\mathbf{J}, \tag{2.1}$$

which is equivalent to

$$G(h)\mathbf{J}'=\mathbf{R}\mathbf{J}, \tag{2.2}$$

where $\mathbf{E}$ is an unit matrix of order 3, $\mathbf{J}=\mathrm{col}(I_0,I_1,I_2)$, and

$$\mathbf{S}=\begin{pmatrix}0 & \frac{1}{3}k(k+1) & \frac{1}{3}(-k^2+k-1)\\ 0 & \frac{1}{3}k(k^2-k+1) & -\frac{1}{3}(k+1)(k-1)^2\\ 0 & \frac{1}{3}k(k+1)(k-1)^2 & \frac{1}{3}(-k^4+k^3+k^2+k-1)\end{pmatrix},$$

$$\mathbf{N}=\begin{pmatrix}3 & 0 & 0\\ -\frac{1}{3}(k+1) & 4 & 0\\ \frac{1}{3}(-k^2+k-1) & -\frac{2}{3}(k+1) & 5\end{pmatrix},$$

$$\mathbf{R}=\begin{pmatrix}a_{00}(h) & a_{01}(h) & a_{02}(h)\\ a_{10}(h) & a_{11}(h) & a_{12}(h)\\ a_{20}(h) & a_{21}(h) & a_{22}(h)\end{pmatrix},$$

$$
\begin{aligned}
G(h) &= 192h\left(h-\frac{-2k+1}{12}\right)\left(h-\frac{k^3(k-2)}{12}\right),\\
a_{00}(h) &= 144h^2+\frac{4}{3}(-10k^4+21k^3-k^2+21k-10)h\\
&\qquad -\frac{4}{3}k^3(2k-1)(k-2),\\
a_{01}(h) &= -\frac{8}{3}(k+1)(k^2+5k+1)h+\frac{14}{9}k^2(k+1)(2k-1)(k-2),\\
a_{02}(h) &= 20(k^2-k+1)h-\frac{5}{3}k^2(k-2)(2k-1),\\
a_{10}(h) &= -16(k+1)h^2+\frac{4}{3}k(k+1)(2k-1)(k-2)h,\\
a_{11}(h) &= 192h^2+\frac{8}{3}(-7k^4+6k^3+8k^2+6k-7)h,\\
a_{12}(h) &= 20(k+1)(k-1)^2h,\\
a_{20}(h) &= 16(-k^2+k-1)h^2+\frac{4}{3}k^2(k-2)(2k-1)h,\\
a_{21}(h) &= -32(k+1)h^2-\frac{8}{3}k(k+1)(7k^2-13k+7)h,\\
a_{22}(h) &= 240h^2+20k(k^2-k+1)h.
\end{aligned}
$$

Proof: It follows from (1.5) that

$$\frac{\partial y}{\partial h}=\frac{1}{y} \tag{2.3}$$

and

$$y\frac{\partial y}{\partial x}=kx-(k+1)x^2+x^3. \tag{2.4}$$

Obviously, (2.3) implies that

$$I_i'(h)=\oint_{\Gamma_h}\frac{x^i}{y}\,dx. \tag{2.5}$$

Mutiplying (2.4) by y and integrating over Γ_h give

$$I_3=-kI_1+(k+1)I_2. \tag{2.6}$$

Use (1.5) and (2.5) to get

$$\begin{aligned} I_i(h) &= \oint_{\Gamma_h} \frac{x^i y^2}{y}\,dx \\ &= \oint_{\Gamma_h} \frac{x^i(2h + kx^2 - \frac{2}{3}(k+1)x^3 + \frac{1}{2}x^4)}{y}\,dx \\ &= 2hI_i' + kI_{i+2}' - \frac{2}{3}(k+1)I_{i+3}' + \frac{1}{2}I_{i+4}'. \end{aligned} \tag{2.7}$$

On the other hand, using (2.3), (2.4) and integrating by parts, we have

$$\begin{aligned} I_i(h) &= -\frac{1}{i+1}\oint_{\Gamma_h} x^{i+1}\,dy \\ &= -\frac{1}{i+1}\oint_{\Gamma_h} \frac{x^{i+1}(kx - (k+1)x^2 + x^3)}{y}\,dx \\ &= -\frac{1}{i+1}(kI_{i+2}' - (k+1)I_{i+3}' + I_{i+4}'). \end{aligned} \tag{2.8}$$

Eliminating I_{i+4}' from (2.7) and (2.8) yields

$$(i+3)I_i = 4hI_i' + kI_{i+2}' - \frac{1}{3}(k+1)I_{i+3}'. \tag{2.9}$$

This gives

$$3I_0 = 4hI_0' + kI_2' - \frac{1}{3}(k+1)I_3', \tag{2.10}$$

$$4I_1 = 4hI_1' + kI_3' - \frac{1}{3}(k+1)I_4', \tag{2.11}$$

$$5I_2 = 4hI_2' + kI_4' - \frac{1}{3}(k+1)I_5'. \tag{2.12}$$

Substituting (2.6) into (2.10), we obtain the first equation of (2.1). The formula (2.8) implies

$$I_{i+4}' = -(i+1)I_i(h) - kI_{i+2}' + (k+1)I_{i+3}'. \tag{2.13}$$

Taking $i = 0$ in (2.13) and using (2.6), the formula (2.11) give the second equation of (2.1).

Repeating the same arguments, we obtain the third equation. The lemma has been proved.

Denote

$$P(h) = \frac{I_1(h)}{I_0(h)}, \quad Q(h) = \frac{I_2(h)}{I_0(h)}, \tag{2.14}$$

where $h \in [\frac{-2k+1}{12}, 0]$.

Proposition 2.2. *$P(h)$, $Q(h)$ are analytic function for $h \in [\frac{-2k+1}{12}, 0)$, and*

i) $$I_i\left(\frac{-2k+1}{12}\right) = 0, \quad I_i(h) > 0, \quad i = 0, 1, 2,$$

ii) $$P\left(\frac{-2k+1}{12}\right) = Q\left(\frac{-2k+1}{12}\right) = 1, \quad P(h) > 0, \quad Q(h) > 0,$$

iii) $$P'\left(\frac{-2k+1}{12}\right) = -\frac{k-2}{2(k-1)^2}, \quad Q'\left(\frac{-2k+1}{12}\right) = -\frac{k-3}{2(k-1)^2},$$

iv) $$P''\left(\frac{-2k+1}{12}\right) = \frac{(k-2)(-257+257k-110k^2)}{72(k-1)^5},$$
$$Q''\left(\frac{-2k+1}{12}\right) = \frac{651-788k+467k^2-110k^3}{72(k-1)^5}.$$

Proof: The results i) and ii) follows from Green's formula. $P(\frac{-2k+1}{12}) = Q(\frac{-2k+1}{12}) = 1$ imply that

$$P(h) = \frac{1+o(h-\frac{-2k+1}{12})}{1+o(h-\frac{-2k+1}{12})}, \quad Q(h) = \frac{1+o(h-\frac{-2k+1}{12})}{1+o(h-\frac{-2k+1}{12})} \tag{2.15}$$

as $h \to \frac{-2k+1}{12}$. Noting $I_i(h)$ is analytic at $h = \frac{-2k+1}{12}$ (see [**15**]) and $I_0(h) > 0$ for $h \in (\frac{-2k+1}{12}, 0)$, the formula (2.15) implies that $P(h)$ and $Q(h)$ are analytic functions for $h \in [\frac{-2k+1}{12}, 0)$.

Using

$$P' = \frac{I_1' I_0 - I_0' I_1}{I_0^2}, \quad Q' = \frac{I_2' I_0 - I_0' I_2}{I_0^2}$$

and system (2.2) give

$$\begin{cases} GP' = a_{10} + (a_{11} - a_{00})P + a_{12}Q - a_{01}P^2 - a_{02}PQ, \\ GQ' = a_{20} + a_{21}P + (a_{22} - a_{00})Q - a_{01}PQ - a_{02}Q^2. \end{cases} \tag{2.16}$$

Differentiating (2.16) once (resp. twice) yields iii) (resp. iv)).

It is well known that $I(h)$ has the expansion near $h = \frac{-2k+1}{12}$ (see [**15**])

$$I(h) = b_1\left(h - \frac{-2k+1}{12}\right) + b_2\left(h - \frac{-2k+1}{12}\right)^2 + \cdots. \tag{2.17}$$

Theorem 2.3.

i)

$$b_1 = (\alpha+\beta+\gamma) I_0' \left(\frac{-2k+1}{12} \right),$$

$$b_2 = -\frac{(k-2)\beta + (k-3)\gamma}{2(k-1)^2} I_0' \left(\frac{-2k+1}{12} \right) \quad \text{if } b_1 = 0,$$

$$b_3 = \frac{5(k-2)\beta}{6(k-1)^3(k-3)} I_0' \left(\frac{-2k+1}{12} \right) \quad \text{if } b_1 = b_2 = 0.$$

ii) *If $b_1 = 0$ (resp. $b_1 = b_2 = 0$), $b_2 \neq 0$ (resp. $b_3 \neq 0$), then there exists one (resp. two) zero of $I(h)$ tend to $h = \frac{-2k+1}{12}$, i.e., system $(1.8)_\epsilon$ has at most one (resp. two) limit cycle tend to $(1,0)$.*

iii) *The conditions $b_1 = b_2 = b_3 = 0$ hold if and only if $I(h) \equiv 0$.*

Proof: (i) It follows from (1.7) and (2.14) that

$$I(h) = I_0(h)(\alpha + \beta P(h) + \gamma Q(h)), \tag{2.18}$$

which gives

$$b_m = \frac{1}{m!} \left\{ \sum_{j=1}^{m} \binom{m}{j} I_0^{(m-j)}(h)[\alpha + \beta P(h) + \gamma Q(h)]^{(j)} \right\} \Bigg|_{h=\frac{-2k+1}{12}}. \tag{2.19}$$

Therefore, the result i) follows from Proposition 2.2 and above equality.

(ii) In a neighbourhood of $(1,0)$, The Poincare map is

$$P(h) = \epsilon I(h) + o(\epsilon),$$

which yields ii).

(iii) The conditions $b_1 = b_2 = b_3 = 0$ hold if and only if

$$\begin{cases} \alpha + \beta + \gamma = 0, \\ (k-2)\beta + (k-3)\gamma = 0, \\ \beta = 0, \end{cases}$$

which implies $\alpha = \beta = \gamma = 0$. Hence, $I(h) \equiv 0$.

Rewrite (1.5) in the form

$$\frac{1}{2}y^2 + \Phi(x) = h, \tag{2.20}$$

where $\Phi(x) = -\frac{1}{2}kx^2 + \frac{1}{3}(k+1)x^3 - \frac{1}{4}x^4$ satisfying

$$\Phi'(x)(x-1) > 0 \quad \text{for} \quad x \in (0,1) \cup (1, x_1). \tag{2.21}$$

For any $x \in (0,1)$, there is an unique $\widetilde{x} \in (1, x_1)$, such that

$$\Phi(x) = \Phi(\widetilde{x}), \quad 0 < x < 1 < \widetilde{x} < x_1. \tag{2.22}$$

Therefore, we can define a function $\widetilde{x} = \widetilde{x}(x)$ for $0 < x < 1$ satisfying (2.22). By (2.21) and (2.22), we have

$$\frac{d\widetilde{x}}{dx} = \frac{\Phi'(x)}{\Phi'(\widetilde{x})} < 0. \tag{2.23}$$

Lemma 2.4. $x_1 < x + \widetilde{x} < 2$, $x\widetilde{x} < 1$.

Proof: Let

$$a = x + \widetilde{x}, \quad \text{and} \quad b = x\widetilde{x}. \tag{2.24}$$

The equality (2.22) implies that

$$\frac{1}{2}ka - \frac{1}{3}(k+1)a^2 + \frac{1}{4}a^3 + b\left[\frac{1}{3}(k+1) - \frac{1}{2}a\right] = 0. \tag{2.25}$$

Taking $a = \frac{2}{3}(k+1)$ into (2.25), we have $-\frac{1}{18}(2k-1)(k-2) = 0$, which contradicts the assumption $k > 2$. This shows $a \neq \frac{2}{3}(k+1)$. Hence

$$b = \frac{6ka - 4(k+1)a^2 + 3a^3}{6a - 4(k+1)}. \tag{2.26}$$

To find the maximal or minimal value of $a(x)$, we consider the equation $\frac{da(x)}{dx} = 0$, which is equivalent to

$$\Phi'(x) + \Phi'(\widetilde{x}) = 0. \tag{2.27}$$

The relationship

$$x^2 + \widetilde{x} = a^2 - 2b$$

and (2.26) yield

$$\Phi'(x) + \Phi'(\widetilde{x}) = \frac{1}{2}a(a-2)(a-2k). \tag{2.28}$$

The inequality $0 < x < 1 < \widetilde{x} < k$ implies $0 < a < 2k$, hence $a = 2$ is the unique root of the equation (2.27). Noting $x \in [0,1]$ and $a(0) = x_1$, $a(1) = 2$, we have

$$x_1 < a = x + \widetilde{x} < 2,$$

which implies that $x\widetilde{x} < x(2-x) \leq 1$.

Near the value $h = 0$ corresponding to a saddle-loop Γ_0, Abelian integral $I(h)$ has the expansion [**14**]

$$I(h) = c_0 + c_1 h ln|h| + c_2 h + \cdots \tag{2.29}$$

with $c_0 = I(0)$, $c_1 = \widetilde{c}\,\mathrm{div}(X,Y)|_{(0,0)}$, $c_2 = I^{'}(0)$ if $c_1 = 0$, where $\widetilde{c}$ is a constant. Using this formula, we obtain

$$\begin{aligned} I_0(h) &= I_0(0) + c_{01} h ln|h| + c_{02} h + \cdots, \\ I_1(h) &= I_1(0) + I_1^{'}(0)h + \cdots, \\ I_2(h) &= I_2(0) + I_2^{'}(0)h + \cdots. \end{aligned} \tag{2.30}$$

It follows from (1.7) and (2.30) that

$$\begin{aligned} c_0 &= \alpha I_0(0) + \beta I_1(0) + \gamma I_2(0), \\ c_1 &= \alpha c_{01}, \quad c_{01} \neq 0, \\ c_2 &= \beta I_1^{'}(0) + \gamma I_2^{'}(0) \quad \text{if} \quad \alpha = 0. \end{aligned} \tag{2.31}$$

Lemma 2.5.
i) $\frac{d}{dh}(\frac{I_2(h)}{I_1(h)}) > 0$, $h \in (\frac{-2k+1}{12}, 0)$,
ii) $I_1(0)I_2'(0) - I_1'(0)I_2(0) > 0$.

Proof: (i) Denote

$$\xi(x) = \frac{x^2 - \widetilde{x}^2 \frac{d\widetilde{x}}{dx}}{x - \widetilde{x}\frac{d\widetilde{x}}{dx}}, \quad x \in (0,1).$$

It follows from (2.23) that

$$\xi(x) = \frac{k - x\widetilde{x}}{k + 1 - x - \widetilde{x}},$$

which gives

$$\xi^{'}(x) = \frac{\mathcal{A}(x,\widetilde{x})}{(k+1-x-\widetilde{x})^2},$$

where

$$\mathcal{A}(x,\widetilde{x}) = -\left(\widetilde{x} + x\frac{d\widetilde{x}}{dx}\right)(k+1-x-\widetilde{x}) + \left(1+\frac{d\widetilde{x}}{dx}\right)(k - x\widetilde{x})$$
$$= \frac{1}{\widetilde{x}(\widetilde{x}-1)(\widetilde{x}-k)}[\widetilde{x}(\widetilde{x}-1)^2(\widetilde{x}-k)^2 + x(x-1)^2(x-k)^2].$$

Noting $0 < x < 1 < \widetilde{x} < k$, we have $\mathcal{A}(x,\widetilde{x}) < 0$, which implies $\xi^{'}(x) < 0$. Use this result and Theorem 1 of [**9**], we get i).

(ii) By symmetry and $\widetilde{x} = \widetilde{x}(x)$, $\widetilde{z} = \widetilde{z}(z)$, $y(\widetilde{x}) = y(x)$, $y(\widetilde{z}) = y(z)$, we get

$$I_1(0)I_2^{'}(0) - I_1^{'}(0)I_2(0)$$
$$= 2\int_0^{x_1} xy\,dx \cdot 2\int_0^{x_1} \frac{z^2}{y(z)}\,dz - 2\int_0^{x_1}\frac{z}{y(z)}\,dz \cdot 2\int_0^{x_1} x^2 y\,dx$$
$$= 2\int_0^{x_1}\int_0^{x_1}\left[\frac{xy(x)z(z-x)}{y(z)} + \frac{xy(z)z(x-z)}{y(x)}\right]dx\,dz$$
$$= 2\int_0^1\int_0^1 \frac{[y^2(x)-y^2(z)]}{y(x)y(z)}\Phi(x,z)\,dx\,dz,$$

where

$$\Phi(x,z) = xz(z-x) - \widetilde{x}z(z-\widetilde{x})\frac{d\widetilde{x}}{dx} - x\widetilde{z}(\widetilde{z}-x)\frac{d\widetilde{z}}{dz} + \widetilde{x}\widetilde{z}(\widetilde{z}-\widetilde{x})\frac{d\widetilde{x}}{dx}\frac{d\widetilde{z}}{dz}$$
$$= \left(x - \widetilde{x}\frac{d\widetilde{x}}{dx}\right)\left(z - \widetilde{z}\frac{d\widetilde{z}}{dz}\right)(\xi(z) - \xi(x)),$$

$\xi(x)$ is defined as above. Since

$$\xi'(x) < 0,\ \frac{d\widetilde{x}}{dx} < 0,\ \frac{d\widetilde{z}}{dz} < 0 \quad \text{and} \quad y^{'}(x) > 0 \quad \text{for} \quad x \in (0,1),$$

we get $I_1(0)I_2^{'}(0) - I_1^{'}(0)I_1(0) > 0$.

Theorem 2.6. i) *If $c_0 = 0$, $c_1 \neq 0$ (resp. $c_0 = c_1 = 0$), then $I(h)$ has at most one (resp. two) zero near $h = 0$, i.e., system (1.8)$_\epsilon$ has at most one (resp. two) limit cycle that tend to the saddle-loop Γ_0 of system (1.4).*

ii) *The condition $c_0 = c_1 = c_2 = 0$ is equivalent to $I(h) \equiv 0$.*

Proof: (i) It follows from Theorem C of [**14**].

(ii) Obviously, $c_0 = c_1 = c_2 = 0$ if and only if

$$\begin{cases} \alpha I_0(0) + \beta I_1(0) + \gamma I_2(0) = 0, \\ \alpha = 0, \\ \beta I_1'(0) + \gamma I_2'(0) = 0. \end{cases}$$

Lemma 2.5 implies that

$$\begin{vmatrix} I_0(0) & I_1(0) & I_2(0) \\ 1 & 0 & 0 \\ 0 & I_1'(0) & I_2'(0) \end{vmatrix} = -[I_1(0)I_2'(0) - I_1'(0)I_2(0)] < 0.$$

Thus, $c_0 = c_1 = c_2 = 0$ if and only if $\alpha = \beta = \gamma = 0$, which implies $I(h) \equiv 0$.

We end this section by several inequalities, which are crucial for our analysis in next two sections.

Lemma 2.7.

i) $I_1(0) < I_2(0) < I_0(0)$,
ii) $(k-2)I_2(0) - (k-3)I_1(0) - I_0(0) < 0$.

Proof: (i) Lemma 2.5 i) and Proposition 2.2 imply that

$$1 = \frac{I_2(\frac{-2k+1}{12})}{I_1(\frac{-2k+1}{12})} < \frac{I_2(h)}{I_1(h)} < \frac{I_2(0)}{I_1(0)}$$

for $h \in (\frac{-2k+1}{12}, 0)$, which gives $I_1(0) < I_2(0)$. On the other hand,

$$\begin{aligned} I_2(0) - I_0(0) &= 2\int_0^1 (x^2-1)y\,dx + 2\int_1^{x_1} (\widetilde{x}-1)y(\widetilde{x})\,d\widetilde{x} \\ &= 2\int_0^1 \frac{y}{\Phi'(\widetilde{x})}(x-1)(\widetilde{x}-1)(\widetilde{x}-x)(k-x-\widetilde{x}-x\widetilde{x})\,dx. \end{aligned} \tag{2.32}$$

It follows from Lemma 2.4 that $k-(x+\widetilde{x}+x\widetilde{x}) \geq k-3 > 0$. Hence, application of (2.32) yields $I_2(0) < I_0(0)$. Summing up the above discussion, we get i).

(ii) Using same arguments as (i), we have

$$\begin{aligned} &(k-2)I_2(0) - (k-3)I_1(0) - I_0(0) \\ &\quad = 2\int_0^1 \frac{y}{\Phi'(\widetilde{x})}(x-1)(\widetilde{x}-1)(\widetilde{x}-x)[k-x-\widetilde{x}-(k-2)x\widetilde{x}]\,dx. \end{aligned}$$

It follows from Lemma 2.4 that $k-x-\widetilde{x}-(k-2)x\widetilde{x} > 0$. Since $\Phi'(x) > 0$ and $0 < x < 1 < \widetilde{x}$, the above equalities gives ii).

3. Behaviour of curve $\omega(h) = \frac{I_1''(h)}{I_0''(h)}$ and relevant results

Lemma 3.1. *For* $h \in (\frac{-2k+1}{12}, 0)$, $I_0''(h) > 0$.

Proof: Chow [**3**] and Gavrilov [**6**] have proved that the period function of (1.4) is monotonic, i.e., $I_0''(h) \neq 0$ for $h \in (\frac{-2k+1}{12}, 0)$. On the other hand, since $I_0'(h) > 0$, the formula (2.30) implies $c_{01} < 0$. This gives

$$I_0''(h) = \frac{c_{01}}{h} + \cdots > 0$$

as $h \to 0^-$, which yields the result.

Define

$$\omega(h) = \frac{I_1''(h)}{I_0''(h)}, \quad h \in \left(\frac{-2k+1}{12}, 0\right). \tag{3.1}$$

In this section, we shall derive the Riccati equation satisfied by $\omega(h)$ and discuss the behaviour of curve $\omega(h)$. The upshot is to prove that $I(h)$ has at most three zeros in $(\frac{-2k+1}{12}, 0)$.

Lemma 3.2.

$$\begin{aligned} I_2''(h) = {} & \frac{-12h}{(2k-1)(k-2)} I_0''(h) \\ & + \frac{1}{k+1}\left[\frac{36h}{(k-2)(2k-1)} + k\right] I_1''(h). \end{aligned} \tag{3.2}$$

Proof: Differentiating both sides of (2.1) yields

$$(4h\mathbf{E} + \mathbf{S})\mathbf{J}'' = (\mathbf{N} - 4\mathbf{E})\mathbf{J}', \tag{3.3}$$

where

$$\mathbf{N} - 4\mathbf{E} = \begin{pmatrix} -1 & 0 & 0 \\ -\frac{1}{3}(k+1) & 0 & 0 \\ \frac{1}{3}(-k^2 + k - 1) & -\frac{2}{3}(k+1) & 1 \end{pmatrix}.$$

Eliminating I_0' from the first two equations of (3.3), we get (3.2).

Lemma 3.3. *The integral I_0, I_1 satisfy the following equation*

$$G(h)\begin{pmatrix} I_0''' \\ I_1''' \end{pmatrix} = \begin{pmatrix} A(h) & B(h) \\ C(h) & D(h) \end{pmatrix}\begin{pmatrix} I_0'' \\ I_1'' \end{pmatrix} \tag{3.4}$$

where

$$A(h) = \frac{48(-7k^2+22k-7)}{(2k-1)(k-2)}h^2 + \frac{4}{3}(14k^4-27k^3-10k^2-27k+14)h$$
$$+\frac{4}{3}k^3(2k-1)(k-2),$$
$$B(h) = \frac{1}{k+1}\left[\frac{-432(k^2-k+1)}{(2k-1)(k-2)}h^2 + \frac{4}{3}(-2k^4+k^3+60k^2+k-2)h\right.$$
$$\left.-\frac{1}{9}k^2(2k-1)(k-2)(10k^2+11k+10)\right],$$
$$C(h) = \frac{16(k+1)(7k^2-13k+7)}{(2k-1)(k-2)}h^2 + \frac{4}{3}k(k+1)(2k-1)(k-2)h,$$
$$D(h) = \frac{-48(17k^2-38k+17)}{(2k-1)(k-2)}h^2 + \frac{4}{3}(10k^4-21k^3+10k^2-21k+10)h.$$

Proof: Differentiate once (3.3), we get

$$(4h\mathbf{E}+\mathbf{S})\mathbf{J}''' = (\mathbf{N}-8\mathbf{E})\mathbf{J}''. \tag{3.5}$$

Substituting (3.2) into the first two equations of (3.5), we get (3.4).

Theorem 3.4. *The ratio $\omega(h)$ satisfies the following Riccati equation*

$$G(h)\omega'(h) = C(h) + (D(h)-A(h))\omega - B(h)\omega^2. \tag{3.6}$$

Proof: Since

$$\omega' = \frac{I_1'''I_0'' - I_1''I_0'''}{(I_0'')^2},$$

the equation (3.6) follows from Lemma 3.3.

Lemma 3.5. *For $h \in (\frac{-2k+1}{12}, 0)$, $B(h) < 0$, $C(h) < 0$, which implies $(D(h)-A(h))^2 + 4B(h)C(h) > 0$.*

Proof: Denote

$$B_1(h) = \frac{4}{3}(-2k^4 + k^3 + 60k^2 + k - 2)h - \frac{1}{9}k^2(2k-1)(k-2)(10k^2 + 11k + 10),$$

$$C_1(h) = \frac{16(k+1)(7k^2 - 13k + 7)}{(2k-1)(k-2)}h + \frac{4}{3}k(k+1)(2k-1)(k-2),$$

which gives

$$B(h) = \frac{1}{k+1}\left[\frac{-432(k^2-k+1)}{(2k-1)(k-2)}h^2 + B_1(h)\right], \quad C(h) = hC_1(h). \tag{3.7}$$

Since $B_1(h)$ is linear function of h and

$$B_1(0) = -\frac{1}{9}k^2(2k-1)(k-2)(10k^2 + 11k + 10) < 0,$$

$$B_1\left(\frac{-2k+1}{12}\right) = -\frac{1}{9}(2k-1)[(k-2)(10k^4 + 9k^3 + 7k^2 + 54k + 109) + 216] < 0,$$

this shows $B_1(h) < 0$. It follows from (3.7) that $B(h) < 0$.

Similarly, we get $C_1(h) > 0$, which implies $C(h) < 0$ for $h \in (\frac{-2k+1}{12}, 0)$.

Proposition 3.6. *For $h \in (\frac{-2k+1}{12}, 0)$, $\omega(h)$ is analytic and*

i) $\omega'(h) > 0$,
ii) $-\frac{(k+1)(2k-7)}{10k^2-31k+31} < \omega(h) < 0$.

Proof: By Theorem 3.4, the curve $\omega(h)$ is the trajectory of system

$$\begin{cases} \dot{h} = G(h), \\ \dot{\omega} = C(h) + (D(h) - A(h))\omega - B(h)\omega^2, \end{cases} \tag{3.8}$$

which has four critical points in $\{(h,\omega) \mid \frac{-2k+1}{12} \le h \le 0\}$: a stable node at $E_1(0,0)$, two saddles at $J_1(\frac{-2k+1}{12}, \frac{(k+1)(2k-7)}{10k^2-31k+31})$ and $E_2(0, \frac{12k(k+1)}{10k^2+11k+10})$, an unstable node at $J_2(\frac{-2k+1}{12}, 1)$. The isocline $\omega^{\pm}(h)$ is determined by algebraic curve

$$C(h) + (D(h) - A(h))\omega - B(h)\omega^2(h) = 0, \tag{3.9}$$

where

$$\omega^+(h) = \frac{D-A-\sqrt{(D-A)^2+4BC}}{2B}, \tag{3.10}$$

$$\omega^-(h) = \frac{D-A+\sqrt{(D-A)^2+4BC}}{2B} \tag{3.11}$$

with

$$\omega^-\left(\frac{-2k+1}{12}\right) = -\frac{(k+1)(2k-7)}{10k^2-31k+31}, \quad \omega^-(0)=0.$$

Differentiating (3.9) once, we have

$$\begin{aligned}(\omega^-)'\left(\frac{-2k+1}{12}\right)& \\ &= \frac{35(k-2)(k+1)(2k-1)(2k^2-11k+11)}{(k-1)^2(10k^2-31k+31)^2} > 0.\end{aligned} \tag{3.12}$$

Assume $\frac{d\omega^-}{dh}=0$ at $h=\overline{h}$ and $(\omega^-)'(h)>0$ for $h\in(\frac{-2k+1}{12},\overline{h})$, which implies $(\omega^-)''(\overline{h})<0$. Differentiate (3.9) twice to get

$$(\omega^-)''(\overline{h}) = \frac{432(k^2-k+1)(\omega^- - \frac{k+1}{3})(\omega^- - \frac{(k+1)(7k^2-13k+7)}{9(k^2-k+1)})}{B(h)(k+1)(2k-1)(k-2)(\omega^- - \frac{D-A}{2B})}. \tag{3.13}$$

By Lemma 3.5 and (3.11), we have $\omega^- - \frac{D-A}{2B} < 0$, $B(\overline{h}) < 0$ and $\omega^-(\overline{h}) < 0$. Therefore, the formula (3.13) gives $(\omega^-)''(\overline{h}) > 0$. This contradicts the assumption, which yields that the isocline $\omega=\omega^-(h)$ is monotonically increasing function for $h\in(\frac{-2k+1}{12},0)$.

Since $I_i(h)$ is analytic at $h=\frac{-2k+1}{12}$, it follows from (3.4) that

$$A\left(\frac{-2k+1}{12}\right)I_0''\left(\frac{-2k+1}{12}\right)+B\left(\frac{-2k+1}{12}\right)I_2''\left(\frac{-2k+1}{12}\right)=0,$$

which implies

$$\omega\left(\frac{-2k+1}{12}\right) = -\frac{(k+1)(2k-7)}{10k^2-31k+31}. \tag{3.14}$$

Lemma 3.1 and (3.14) show that $\omega(h)$ is analytic for $h\in[\frac{-2k+1}{12},0)$. On the other hand, the formula (2.30) gives

$$\omega(0) = \lim_{h\to 0}\frac{I_1''}{I_0''} = 0. \tag{3.15}$$

Hence, $\omega(h)$ is the trajectory of (3.13) from J_1 to E_1. Since $(\omega^-)'(h) > 0$, the graph of $\omega(h) = \frac{I_1''}{I_0''}$ must stay in the region $\{(h, \omega) \mid \omega < \omega^-,\ h \in (\frac{-2k+1}{12}, 0)\}$, which implies $\omega'(h) > 0$, see Figure 3.1. The inequality ii) follows from i), (3.14) and (3.15).

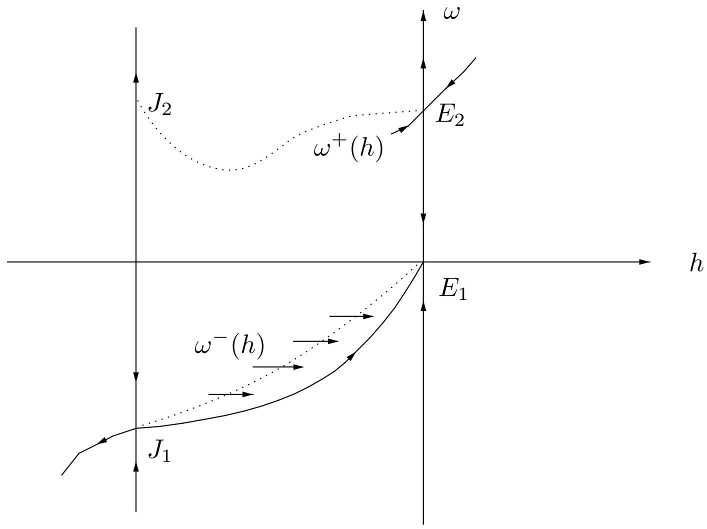

Figure 3.1

Corollary 3.7. i) *If $\alpha + \frac{1}{3}(\gamma k + \beta k + \beta) = 0$, $\gamma \neq 0$, then $I''(h)$ has $h = h^* = -\frac{(\gamma k+\beta k+\beta)(k-2)(2k-1)}{36\gamma}$ as the unique zero in $(\frac{-2k+1}{12}, 0)$. If $\alpha + \frac{1}{3}(\gamma k + \beta k + \beta) \neq 0$, then $h = h^*$ is not the zero of $I''(h)$.*

ii) *If $\alpha + \frac{1}{3}(\gamma k + \beta k + \beta) > 0$, $-\frac{k-2}{k-3}\beta < \gamma < 0$, then $I''(h)$ has at most one zero in $(\frac{-2k+1}{12}, 0)$.*

iii) *$P'(h) < 0$ for $h \in (\frac{-2k+1}{12}, 0)$.*

Proof: (i) Lemma 3.2 yields

$$(3.16) \quad I''(h) = \frac{(2k-1)(k-2)\alpha - 12\gamma h}{(2k-1)(k-2)} I_0'' + \frac{36\gamma h + (\gamma k + \beta k + \beta)(2k-1)(k-2)}{(k+1)(2k-1)(k-2)} I_1''.$$

If $\alpha + \frac{1}{3}(\gamma k + \beta k + \beta) = 0$, then

$$I''(h) = \frac{36\gamma I_0''(h)(\omega(h) - \frac{k+1}{3})(h - h^*)}{(k+1)(2k-1)(k-2)},$$

which implies that $I''(h)$ has $h = h^*$ as the unique zero. If $\alpha + \frac{1}{3}(\gamma k + \beta k + \beta) \neq 0$, then it follows from Lemma 3.1 and (3.16) that

$$I''(h^*) = \left[\alpha + \frac{1}{3}(\gamma k + \beta k + \beta)\right] I_0''(h^*) \neq 0.$$

(ii) In the case of $\alpha + \frac{1}{3}(\gamma k + \beta k + \beta) > 0$, $-\frac{k-2}{k-3}\beta < \gamma < 0$, $h = h^*$ is not a zero of $I''(h)$, and (3.16) is equivalent to

$$I''(h) = \frac{36\gamma(h - h^*)}{(k+1)(2k-1)(k-2)} I_0''(h)q(h), \tag{3.17}$$

where

$$q(h) = f(h) + \omega(h)$$

and

$$f(h) = \frac{(k+1)[(2k-1)(k-2)\alpha - 12\gamma h]}{36\gamma(h - h^*)}, \tag{3.18}$$

which implies

$$f'(h) = -\frac{(k+1)(2k-1)(k-2)[\alpha + \frac{1}{3}(\gamma k + \beta k + \beta)]}{36\gamma(h - h^*)^2} > 0. \tag{3.19}$$

Therefore, by Proposition 3.6, we have

$$q'(h) = f'(h) + \omega'(h) > 0. \tag{3.20}$$

If $-\frac{k+1}{k}\beta \leq \gamma < 0$, then $h^* \geq 0$. This and (3.20) imply that $q(h)$ (i.e., $I''(h)$) has at most one zero in $h \in (\frac{-2k+1}{12}, 0)$.

On the other hand, if $-\frac{k-2}{k-3}\beta < \gamma < -\frac{k+1}{k}\beta < 0$, then $h^* \in (\frac{-2k+1}{12}, 0)$, $\alpha > -\frac{1}{3}(\gamma k + \beta k + \beta)$. The inequality (3.19) gives

$$f(h) < f(0) = \frac{(k+1)\alpha}{\gamma k + \beta k + \beta} < 0$$

for $h \in (h^*, 0)$. Hence, Proposition 3.6 yields $q(h) < 0$ for $h^* < h < 0$. It follows from (3.20) that $q(h)$ has at most one zero in $h \in (\frac{-2k+1}{12}, h^*)$. We obtain ii) by using i), Lemma 3.1 and (3.17).

(iii) Consider the Abelian integral

$$I(h) = \alpha I_0(h) + I_1(h) = I_0(h)(\alpha + P(h)).$$

If $\alpha > 0$, then $I(h) > 0$.

If $\alpha < 0$, then Lemma 3.1 and Proposition 3.6 show that

$$I''(h) = I_0''(h)(\alpha + \omega(h)) < 0,$$

which implies the curve $I(h)$ is concave for $h \in (\frac{-2k+1}{12}, 0)$. Therefore, noticing $I(\frac{-2k+1}{12}) = 0$ and $I_0(h) \neq 0$, $\alpha + P(h)$ has at most one zero for arbitrary constant α. This yields $P(h)$ is monotonic for $h \in (\frac{-2k+1}{12}, 0)$. Suppose $h = h_1$ is the zero of $I(h)$, the convexity of $I(h)$ implies $I'(h_1) = I_0(h_1)P'(h_1) \leq 0$, i.e. $P'(h_1) \leq 0$. However, if $P'(h_1) = 0$, then

$$\begin{aligned} I''(h_1) &= I_0''(h_1)(\alpha + P(h_1)) + 2I_0'(h_1)P'(h_1) + I_0(h_1)P''(h_1) \\ &= I_0(h_1)P''(h_1) < 0, \end{aligned}$$

which shows $P''(h_1) < 0$, i.e., $h = h_1$ is the maximum point of $P(h)$. This contradicts $P'(h_1) \leq 0$. The proof is finished.

Proposition 3.8. *$\omega''(h) > 0$ for $h \in (\frac{-2k+1}{12}, 0)$.*

Proof: We split the proof by several steps.

1) First, $V(h, \omega) = 2D' - 2A' - G'' - 4B'\omega > 0$.

It is readily seen

$$\begin{aligned} V(h, 0) = &-\frac{384}{(2k-1)(k-2)}[(k-2)(11k-1)+9]h \\ &+ \frac{16}{3}(4k^2 - k + 4)(k-1)^2 > 0, \\ V\left(h, -\frac{(2k-7)(k+1)}{10k^2 - 31k + 31}\right) = &-\frac{384(55k^2 - 139k + 139)h}{10k^2 - 31k + 31} \\ &+ \frac{16}{3(10k^2 - 31k + 31)}[k^3(k-4)(40k^2 - 58k + 287) \\ &+ 582k^3 + (211k^2 - 414k) + 138] > 0. \end{aligned} \tag{3.21}$$

Since $V(h, \omega)$ is linear function of ω and $\omega'(h) > 0$, $-\frac{(k+1)(2k-7)}{10k^2-31k+31} < \omega < 0$ (see Proposition 3.6), it follows from (3.21) that $V(h, \omega) > 0$ for $h \in (\frac{-2k+1}{12}, 0)$.

2) If $h = h_1$ satisfies $\omega''(h_1) = 0$, then $\omega'''(h_1) > 0$.

Indeed, differentiate (3.6) twice to get

$$G(h_1)\omega'''(h_1) = C''(h_1) + (D''(h_1) - A''(h_1))\omega(h_1) - B''\omega^2(h_1) \\ + V(h_1, \omega(h_1))\omega'(h_1) - 2B(h_1)(\omega')^2(h_1). \tag{3.22}$$

By Lemma 3.5, Proposition 3.6 and step 1),we conclude that

$$C''(h_1) + (D''(h_1) - A''(h_1))\omega(h_1) - B''(h_1)\omega^2(h_1) \\ = \frac{864(k^2 - k + 1)}{(k+1)(2k-1)(k-2)}\left(\omega(h_1) - \frac{k+1}{3}\right) \\ \times \left(\omega(h_1) - \frac{(k+1)(7k^2 - 13k + 7)}{9(k^2 - k + 1)}\right) > 0,$$

$$V(h_1, \omega(h_1))\omega'(h_1) > 0, \quad -2B(h_1)(\omega')^2(h_1) > 0. \tag{3.23}$$

Hence, the formulas (3.22) and (3.23) imply $\omega'''(h_1) > 0$.

3) $\omega''(\frac{-2k+1}{12}) > 0$.

To prove it, differentiating (3.6) twice, we get

$$\omega''\left(\frac{-2k+1}{12}\right) = \frac{35(k-2)(k+1)(2k-1)}{72(k-1)^5(10k^2 - 31k + 31)^3}g(k), \tag{3.24}$$

where

$$g(k) = 2200k^6 - 24924k^5 + 129246k^4 - 375481k^3 \\ + 604833k^2 - 500511k + 166837.$$

This gives $g^{(i)}(4) > 0$, $i = 0, 1, 2, \dots, 6$, which implies

$$g(k) = \sum_{i=0}^{6} \frac{g^{(i)}(4)}{i!}(k-4)^i > 0, \quad k \in (4, +\infty). \tag{3.25}$$

Hence, the result $\omega''(\frac{-2k+1}{12}) > 0$ follows from (3.24) and (3.25).

4) Finally, we prove $\omega''(h) > 0$.

By step 3), starting from $h = \frac{-2k+1}{12}$, if $h = h_1$ is the *first* point satisfying $\omega''(h_1) = 0$, then $\omega'''(h_1) \le 0$, which contradicts the result proved in step 2). This implies that $\omega''(h)$ has no zero. Therefore, $\omega''(h) > 0$.

Theorem 3.9. *$I(h)$ has at most three zeros (counted with their multiplicities) inside the interval $(\frac{-2k+1}{12}, 0)$.*

Proof: This theorem is proved by several parts.

1) We are going to prove that $I''(h)$ has at most two zeros (counted with their multiplicities), i.e., $I(h)$ has at most two inflection points. Since $I(\frac{-2k+1}{12}) = 0$, this result implies that the maximum number of zeros of $I(h)$ is at most three on the interval $(\frac{-2k+1}{12}, 0)$.

It has been proved in Proposition 3.6 that $\omega'(h) > 0$. Therefore, we can take ω as a new parameter and consider the curve $\nu = \nu(h(\omega))$, defined by

$$\widetilde{\Omega} = \left\{(\omega, \nu) \mid \omega = \omega(h),\, \nu = \nu(h) = \frac{I_2''}{I_0''},\, h \in \left(\frac{-2k+1}{12}, 0\right)\right\} \tag{3.26}$$

where $h = h(\omega)$ is the inverse function of $\omega = \omega(h)$. It is easy to get that

$$\begin{aligned} I''(h) &= I_0''(h)(\alpha + \beta\omega(h) + \gamma\nu(h)), \\ I'''(h) &= I_0''(h)(\beta\omega' + \gamma\nu') && \text{if} \quad I''(h) = 0, \\ I^{(4)}(h) &= I_0''(h)(\beta\omega'' + \gamma\nu'') && \text{if} \quad I''(h) = I'''(h) = 0, \end{aligned}$$

which implies that $\widetilde{\Omega}$ has the following properties:

i) The intersection points of the lines $l\colon \alpha + \beta\omega + \gamma\nu = 0$ with the curve $\widetilde{\Omega}$ in $\omega\nu$-plane correspond to the zeros of $I''(h)$.

ii) $I''(h_0) = I'''(h_0) = 0$ hold if and only if l is tangent to the $\widetilde{\Omega}$ at the point $(\omega(h_0), \nu(h_0))$.

iii) If $(\nu''\omega' - \nu'\gamma'')|_{h=h_0} \neq 0$, then $I''(h_0) = I'''(h_0) = I^{(4)}(h_0) = 0$ hold if and only if $\alpha = \beta = \gamma = 0$, i.e, $I(h) \equiv 0$.

Lemma 3.2 gives

$$\nu(h) = -\frac{12h}{(2k-1)(k-2)} + \left[\frac{36h}{(k+1)(2k-1)(k-2)} + \frac{k}{k+1}\right]\omega,$$

which yields

$$\nu''\omega' - \omega''\nu' = \frac{12}{(k+1)(k-2)(2k-1)}[6(\omega')^2 + (k+1-3\omega)\omega''].\tag{3.27}$$

It follows from Proposition 3.6, Proposition 3.8 and (3.27) that

$$\frac{d^2\nu}{d\omega^2} = \frac{\nu''\omega' - \omega''\nu'}{(\omega')^3} > 0.$$

This implies that $\widetilde{\Omega}$ is convex in $\omega\nu$-plane. Therefore, the maximum number of intersection points of the line $l\colon \alpha + \beta\omega + \gamma\nu = 0$ with $\widetilde{\Omega}$ is at most two. By the properties i)–iii) of $\widetilde{\Omega}$, $I''(h)$ has at most two zeros (counted with their multiplicities).

2) The multiplicity of zero of $I(h)$ is at most three. If $h = h_0$ is the zero of multiplicity 3, then $h = h_0$ is an unique zero of $I(h)$.

Otherwise, suppose the multiplicity of $h = h_0$ is great than 3, i.e., $I(h_0) = I'(h_0) = I''(h_0) = I'''(h_0)$. By step 1), $I''(h)$ has at most two zeros (counted with their multiplicities) in $h \in (\frac{-2k+1}{12}, 0)$, which implies $I^{(4)}(h_0) \neq 0$. Without loss of generality, assume $I^{(4)}(h) > 0$. Hence, $I(h)$ is convex in the neighbourhood of $h = h_0$. Noting $I(\frac{-2k+1}{12}) = 0$, there must exist one inflection point $h = h_1$, $h_1 \in (\frac{-2k+1}{12}, h_0)$, see Figure 3.2(a). Thus, $I''(h)$ has two zeros, one is simple and another is multiplicity two. This contradicts the conclusion proved in step 1).

Suppose $h = h_0$ is the zero with multiplicity 3, i.e., $I(h_0) = I'(h_0) = I''(h_0) = 0$, $I'''(h_0) \neq 0$. Without loss of generality, assume $I'''(h_0) > 0$. Hence, the graph of $I(h)$ is convex for $h > h_0$ and concave for $h < h_0$, $|h - h_0| \ll 0$. Since $I(\frac{-2k+1}{12}) = 0$, $I(h)$ has another inflection point $h = h_1$ between $h = \frac{-2k+1}{12}$ and $h = h_0$, see Figure 3.2(b). By step 1), $I(h)$ has no other inflection point except $h = h_i$, $i = 0, 1$, which implies $h = h_0$ is an unique zero of $I(h)$.

3) If $h = h_0$ is the zero of multiplicity two of $I(h)$, then another zero $h = h_1$ (if there exists) must be simple.

Obviously, $h = h_0$ satisfies $I(h_0) = I'(h_0) = 0$, $I''(h_0) \neq 0$. Without loss of generality, suppose $I''(h_0) > 0$, i.e., $h = h_0$ is minimal point of $I(h)$. Suppose $h_1 > h_0$. Then there must exist two inflection points between $\frac{-2k+1}{12}$ and h_1. Hence, it follows from step 1) that $h = h_1$ must be simple zero of $I(h)$. In the case of $h_1 < h_0$, we can get the result by the same arguments as above.

Summing up above discussion, we get the theorem.

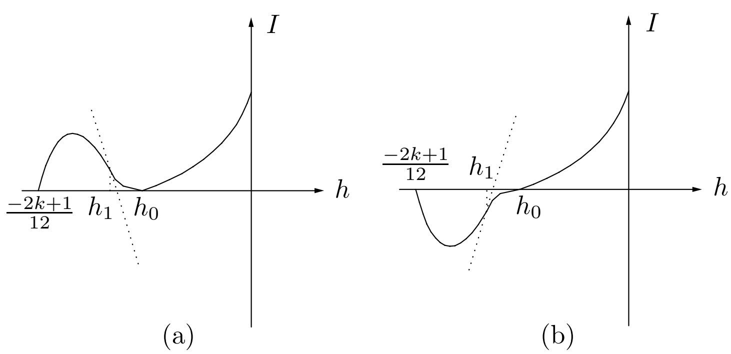

Figure 3.2

4. The geometry of the centriod curve

Definition 4.1. In PQ-plane, the curve

$$\Omega = \left\{ (P, Q) \mid P = P(h), \quad Q = Q(h), \quad h \in \left[\frac{-2k+1}{12}, 0 \right] \right\} \tag{4.1}$$

is called centroid curve.

It has been proved in Corollary 3.7 that $P'(h) < 0$. Therefore, P can be taken as a new parameter and denote Ω as

$$Q = Q(h(p)),$$

where $h(P)$ is the inverse function of $P = P(h)$.

The importance of concept of the centroid curve lies in the fact that its geometry contains the complete information of $I(h)$ although the definition of Ω depends only on $H(x, y) = h$.

From this section, denoted by L_s and by L_c the tangents to Ω at $(P(0), Q(0))$ and $(1, 1)$, i.e., at the endpoints of Ω. L denotes the line $\alpha + \beta P + \gamma Q = 0$, $|\beta| + |\gamma| \neq 0$.

Using same arguments as [**4**], we have

Theorem 4.2. i) *For any $h_0 \in (\frac{-2k+1}{12}, 0)$, the equality $I(h_0) = 0$ holds if and only if the line L passes through the point $(P(h_0), Q(h_0))$.*

ii) *The equalities $I(h_0) = I'(h_0) = 0$ hold if and only if L is tangent to the centroid curve Ω at the point $(P(h_0), Q(h_0))$.*

iii) *If $I(h_0) = I'(h_0) = 0$, then $I''(h_0) = 0$ holds if and only if $P'(h_0)Q''(h_0) - P''(h_0)Q'(h_0) = 0$, i.e., the curvature of Ω at $(P(h_0), Q(h_0))$ is zero.*

Proof: (i) Part i) of the statement follows from (2.19).

(ii) The equation of the tangent line is

$$Q'(h_0)P - P'(h_0)Q + Q(h_0)P'(h_0) - Q'(h_0)P(h_0) = 0. \tag{4.2}$$

By (2.19), $I(h_0) = I'(h_0) = 0$ is equivalent to

$$\begin{cases} \alpha + \beta P(h_0) + \gamma Q(h_0) = 0, \\ \beta P'(h_0) + \gamma Q'(h_0) = 0. \end{cases} \tag{4.3}$$

Solving (4.3) for α and β, we obtain that

$$\alpha = \frac{P(h_0)Q'(h_0) - P'(h_0)Q(h_0)}{P'(h_0)}\gamma, \quad \beta = -\frac{Q'(h_0)}{P'(h_0)}\gamma.$$

Hence, the equation $L\colon \alpha + \beta P + \gamma Q = 0$ is (4.2).

(iii) The condition $I''(h_0) = 0$ when $I(h_0) = I'(h_0) = 0$ is equivalent to

$$\beta P''(h_0) + \gamma Q''(h_0) = 0.$$

This and (4.3) imply the result.

Theorem 4.3. i) *The equation of L_c is*

$$-1 - (k-3)P + (k-2)Q = 0. \tag{4.4}$$

ii) *The coefficient $b_1 = 0$ if and only if L passes through* $(1,1)$.
iii) *The conditions $b_1 = b_2 = 0$ hold if and only if $L = L_c$, where b_1 and b_2 are defined as Theorem 2.3.*

Proof: (i) Part i) of the statement follows from Proposition 2.2.

(ii) By Theorem 2.3, $b_1 = I_0'(\frac{-2k+1}{12})(\alpha+\beta+\gamma)$, which implies ii).

(iii) Theorem 2.3 shows that $b_1 = b_2 = 0$ if and only if

$$\begin{cases} \alpha + \beta + \gamma = 0, \\ (k-2)\beta + (k-3)\gamma = 0. \end{cases}$$

Solving this system for α and β, we obtain $\alpha = -\frac{1}{k-2}\gamma$, $\beta = -\frac{k-3}{k-2}\gamma$, which implies that the equation of L is (4.4).

Theorem 4.4. i) *The equation of L_s is*

$$\frac{Q}{P} = \frac{Q(0)}{P(0)}. \tag{4.5}$$

ii) *The coefficient c_0 is zero if and only if L passes through* $(P(0), Q(0))$.
iii) *The coefficient $c_0 = c_1 = 0$ is equivalent to $L = L_s$, where c_0, c_1 is defined as (2.31).*

Proof: (i) By (2.30) and Lemma 3.1, $\lim_{h\to 0} I_0'(h) = +\infty$, $\lim_{h\to 0} I_1'(h) = I_1'(0)$, $\lim_{h\to 0} I_2'(h) = I_2'(0)$, $\lim_{h\to 0} I_i(h) = I_i(0)$, $i = 0,1,2$. Therefore,

$$\left.\frac{dQ}{dP}\right|_{h=0} = \left.\frac{dQ}{dh}\frac{dh}{dP}\right|_{h=0} = \lim_{h\to 0}\frac{I_2'I_0 - I_0'I_2}{I_1'I_0 - I_0'I_1} = \lim_{h\to 0}\frac{\frac{I_2'I_0}{I_0'} - I_2}{\frac{I_1'I_0}{I_0'} - I_1} = \frac{Q(0)}{P(0)},$$

which yields that the equation of L_s is (4.5).

(ii) By (2.31), the condition $c_0 = 0$ is equivalent to $\alpha + \beta P(0) + \gamma Q(0) = 0$, which implies ii).

(iii) It follows from (2.31) that $c_0 = c_1 = 0$ if and only if

$$\begin{cases} \alpha + \beta P(0) + \gamma Q(0) = 0, \\ \alpha = 0, \end{cases}$$

which implies $\alpha = 0$, $\beta = -\frac{Q(0)}{P(0)}\gamma$. Therefore, the equation of L is (4.5). The result follows.

Proposition 4.5. *L_{cs} doesn't intersect Ω for $h \in (\frac{-2k+1}{12}, 0)$, where L_{cs} is the line passing through both $(1,1)$ and $(P(0), Q(0))$.*

Proof: By the definition of L_{cs} and Theorem 4.3, Theorem 4.4, we have

$$\begin{cases} \alpha + \beta + \gamma = 0, \\ \alpha I_0(0) + \beta I_1(0) + \gamma I_2(0) = 0, \end{cases}$$

which implies

$$\alpha = \frac{I_1(0) - I_2(0)}{I_2(0) - I_0(0)}\beta, \quad \gamma = \frac{I_0(0) - I_1(0)}{I_2(0) - I_0(0)}\beta. \tag{4.6}$$

If $\beta = 0$, then $\gamma = 0$, which contradicts the assumption $|\beta| + |\gamma| \neq 0$. Without loss of generality, suppose $\beta > 0$. The formula (4.6) and Lemma 2.7 give that $\gamma < 0$ and

$$\begin{aligned} \alpha + \frac{1}{3}(\gamma k + \beta k + \beta) &= \frac{\beta}{3[I_2(0) - I_1(0)]} \\ &\qquad [(k-2)I_2(0) - (k-3)I_1(0) - I_0(0)] > 0, \\ \gamma + \frac{k-2}{k-3}\beta &= \frac{\beta}{(k-3)[I_2(0) - I_0(0)]} \\ &\qquad [(k-2)I_2(0) - (k-3)I_1(0) - I_0(0)] > 0. \end{aligned}$$

Corollary 3.7 yields that $I(h)$ has at most one inflection point. Since $I(0) = I(\frac{-2k+1}{12}) = I'(\frac{-2k+1}{12}) = 0$ (cf. Theorem 2.3), $I(h)$ has no zero in $(\frac{-2k+1}{12}, 0)$. The result follows from Theorem 4.2.

Proposition 4.6. i) L_s *doesn't intersect* Ω *for* $h \in [\frac{-2k+1}{12}, 0)$.

ii) L_c *doesn't intersect* Ω *except the endpoint* $(1,1)$.

iii) *The centroid curve* Ω *is concave near its endpoints* $(1,1)$ *and* $(P(0), Q(0))$.

Proof: (i) Denoted by $Q(h)$ and by Q the ordinates of the points on Ω and L_s respectively. By Theorem 4.4 i), we have

$$Q(h) - Q = Q(h) - \frac{Q(0)}{P(0)} P(h) = P(h) \left[\frac{I_2(h)}{I_1(h)} - \frac{I_2(0)}{I_1(0)} \right].$$

Lemma 2.5 implies that $\frac{I_2(h)}{I_1(h)} < \frac{I_2(0)}{I_1(0)}$ for $h \in (\frac{-2k+1}{12}, 0)$, which yields $Q(h) < Q$, i.e., L_s doesn't intersect Ω except $h = 0$.

(ii) By Theorem 4.3 iii), L_c is tangent to Ω at $(1,1)$ if and only if $b_1 = b_2 = 0$, i.e., $\gamma = -\frac{k-2}{k-3}$, $\alpha = \frac{\beta}{k-3}$. This gives $\alpha + \frac{1}{3}(\gamma k + \beta k + \beta) = 0$. It follows from Corollary 3.7 that $h = h^* = \frac{-2k+1}{12}$ is an unique zero of $I''(h)$, which shows that $I(h)$ has no inflection point for $h \in (\frac{-2k+1}{12}, 0)$. Since the curve $I(h)$ is tangent to h-axis at $h = \frac{-2k+1}{12}$ (cf. Theorem 2.3 and Theorem 4.3) and $I(\frac{-2k+1}{12}) = 0$, $I(h)$ has no zero in the interval $(\frac{-2k+1}{12}, 0)$. By Theorem 4.2, L_c does not intersect Ω for $h \in (\frac{-2k+1}{12}, 0)$. Since Proposition 4.5 shows that L_c doesn't pass through $(P(0), Q(0))$, we get ii).

(iii) Proposition 2.2 gives

$$\left.\frac{d^2Q}{dP^2}\right|_{(1,1)} = \left.\frac{Q''P' - P''Q'}{(P')^3}\right|_{h=\frac{-2k+1}{12}} = -\frac{20(k-1)}{3(k-2)^2} < 0,$$

which shows that Ω is concave near the endpoint $(1,1)$.

From (2.30), near $h = 0$, we have

$$\frac{d^2Q}{dP^2} = \frac{1}{(P'(h))^3} \left\{ \frac{c_{01}}{h I_0^3(0)} (I_1(0) I_2'(0) - I_1'(0) I_2(0) + o(h^{-1}) \right\}.$$

In the proof of Lemma 3.1, one gets $c_{01} < 0$. It follows from Lemma 2.5 and Lemma 2.7 iii) that $\frac{d^2Q}{dP^2} < 0$ as $h \to 0^-$, i.e., Ω is cancave near the endpoint $(P(0), Q(0))$.

The analysis we have done shows that

Corollary 4.7. *The centriod curve Ω is entirely placed in the triangle formed by L_s, L_c and L_{cs}, see Figure 4.1.*

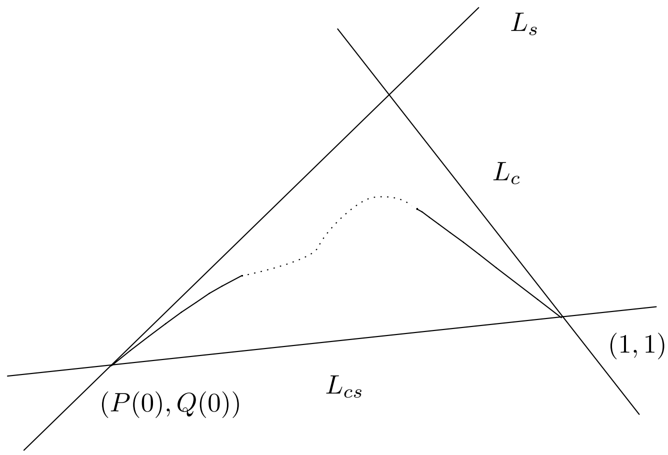

Figure 4.1

5. Proof of main theorem

Theorem 4.2–4.4 reduce the proof of Theorem 1.1 to showing that each line $L\colon \alpha+\beta P+\gamma Q=0$ intersects the centriod curve Ω in at most two points, which implies Ω is a strictly concave curve.

As a sequence of Theorem 3.9 and Theorem 4.2, the following assertion holds:

Lemma 5.1. *If the line L does not pass through $(1,1)$ or $(P(0),Q(0))$, then L intersects Ω in at most three points (counted with their multiplicities).*

Lemma 5.2. *Each line L intersects the centriod curve Ω in at most two points (counted with their multiplicities).*

Proof: We split the proof in several steps.

1) Each line L, passing through $(1,1)$ or $(P(0),Q(0))$, intersects Ω in at most two points (counted with their multiplicities).

For $L=L_c$, L_s or L_{cs}, we have proved the conclusion in Proposition 4.5 and Proposition 4.6. Suppose now that L is a line through $(1,1)$, $L\neq L_c$, $L\neq L_{cs}$, which has another common point M with Ω,

$M \neq (1,1)$ and $(P(0), Q(0))$. Then obviously the points of Ω near $(1,1)$ and those near $(P(0), Q(0))$ lie on different side of L (cf. Corollary 4.7), which yields that either L has no other common point with Ω than $(1,1)$ and M (M is simple), or the total number of intersection points is at least 3 except $(1,1)$ (see Figure 5.1(a)). Now we prove the latter case is impossible.

Indeed, by the conclusion proved in step 1) of the proof of Theorem 3.9, we know that $I(h)$ has at most two inflection points in $(\frac{-2k+1}{12}, 0)$. Since $I(\frac{-2k+1}{12}) = I'(\frac{-2k+1}{12}) = 0$ when L passes through $(1,1)$ (cf. Theorem 2.3 and Theorem 4.3), $I(h)$ has at most two zeros except $h = \frac{-2k+1}{12}$, see Figure 5.1(b), i.e., L has at most two common points with Ω except $(1,1)$, which contradicts the latter case.

If L is a line through $(P(0), Q(0))$ and $L \neq L_s$, $L \neq L_{cs}$, then $I(\frac{-2k+1}{12}) = I(0) = 0$. Using the result proved in step 1) of the proof of Theorem 3.9 again, we have that $I(h)$ has at most two zeros except $h = 0$ and $h = \frac{-2k+1}{12}$. Using the same arguments as above, we get that L intersects Ω in at most two points including $(P(0), Q(0))$.

2) Each tangent $L(h)$, $h \in (\frac{-2k+1}{12}, 0)$, to Ω at point $(P(h), Q(h))$ has exactly one common double point with Ω (the point of tangence).

Indeed, starting from $(P(0), Q(0))$, suppose that $M_0 = (P(h_0), Q(h_0))$, $h_0 \in (\frac{-2k+1}{12}, 0)$ is the *first* point for which $L(h_0)$ has another common point M_1 with Ω (i.e., $M_1 = (P(h_1), Q(h_1)$, $h \neq h_0$). By the result proved in step 1), M_1 doesn't coincide with $(1,1)$ and $(P(0), Q(0))$. The choice of M_1 being the *first* such point implies $L(h_0)$ is tangent to Ω also at M_1 (see Figure 5.2), which contradicts Lemma 5.1. Consequently, there is no $h_1 \in [\frac{-2k+1}{12}, 0]$, for which $L(h_0)$ has another common point with Ω except the tangency point. To prove that is a double intersection point, assume the contrary. Then by Lemma 5.1 the point $(P(h_0), Q(h_0))$ is a triple point of intersection. Theorem 4.2 yields $I(h_0) = I'(h_0) = I''(h_0) = 0$. Slightly moving the tangent $L(h_0)$, we find suitable h_1, h_2 near h_0, for which $I(h_1) = I'(h_1) = 0$, $I(h_2) = 0$. Then accordingly to Theorem 4.2, $L(h_1)$ is tangent to Ω at $(P(h_1), Q(h_1))$, which intersects Ω in another point $(P(h_2), Q(h_2))$. This contradicts the fact we proved above.

3) Suppose that L is not a tangent to Ω at any point, $L \neq L_c$, L_s and L_{cs}. By step 2), Ω is placed entirely on one side of each of its tangents, otherwise the number of the intersection points would be at least 3. This implies Ω is strictly concave. Therefore, L intersects Ω in at most two simple points. Lemma is proved.

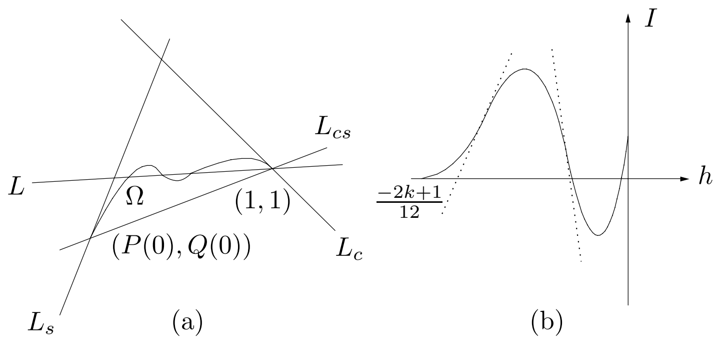

Figure 5.1

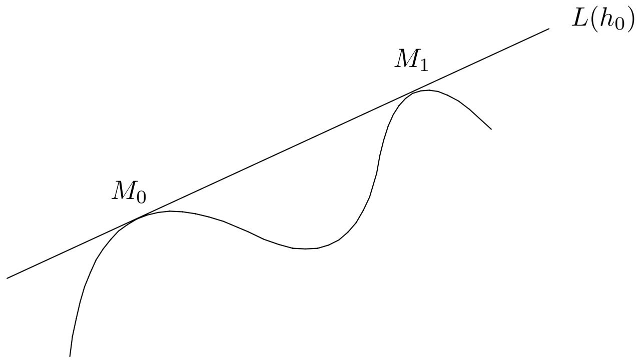

Figure 5.2

Proof of Theorem 1.1: For a given perturbation $(1.8)_\epsilon$, if $\beta = \gamma = 0$, then either the divergenve in $(1.8)_\epsilon$ vanishes identically or it is nowhere zero. In the first case, $(1.8)_\epsilon$ is a Hamiltonian system and in second one no limit cycle can appear in $(1.8)_\epsilon$. Suppose $|\beta| + |\gamma| \neq 0$, which means that the line L is defined. By Lemma 5.2, Theorem 2.3, Theorem 2.6 and Theorem 4.2–4.4, the theorem follows.

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